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Non-resonant interacting ion acoustic waves in a magnetized plasma

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Abstract. We perform an analytical and numerical investigation of the interaction among non-resonant ion acoustic waves in a magnetized plasma. Waves are supposed to be non-resonant, i.e. with different group velocities that are not close to each other. We use an asymptotic perturbation method, based on Fourier expansion and spatio-temporal rescaling. We show that the amplitude slow modulation of Fourier modes cannot be described by the usual nonlinear Schrödinger equation but by a new model system of nonlinear evolution equations. This system is C-integrable, i.e. it can be linearized through an appropriate transformation of the dependent and independent variables. We demonstrate that a subclass of solutions gives rise to envelope solitons. Each envelope soliton propagates with its own group velocity. During a collision solitons maintain their shape, the only change being a phase shift. Numerical results are used to check the validity of the asymptotic perturbation method.

1. Introduction

It is well known that the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\frac{\partial\psi}{\partial\tau} + P(K)\frac{\partial^2\psi}{\partial\xi^2} \pm 2Q(K)|\psi|^2\psi = 0 \tag{1a}$$

describes the slow and small amplitude modulation of a single monochromatic plane wave of wavenumber K. In equation (1a) $\psi(\xi,\tau)$ is the complex amplitude of the monochromatic wave, P and Q are constant coefficients depending on K and the stretched variables ξ, τ are connected to the physical coordinates through the reference frame change ($\varepsilon \ll 1$):

$$\xi = \varepsilon(x - Vt) \qquad \tau = \varepsilon^2 t \tag{1b}$$

where $V = V(K)$ is the group velocity of the wavepacket peaked at wavenumber K. If we consider the plus sign in (1a) and go back to the physical reference frame, envelope soliton solutions can be obtained:

$$\psi(x,t) = \frac{2B}{ch(2B((x - Vt) - 8ABt))}\exp 2\mathrm{i}(A(x - Vt) + (B^2 - A^2)t) \tag{1c}$$

where A, B are real constants.

For the consistency of the approximation method $A, B \ll 1$ and then all solitons travel with about the same velocity $V(K)$ (the group velocity). We conclude that the NLS equation can describe soliton collisions only if solitons are strongly resonant, i.e. if they have very similar velocities, because equation (1a) is obtained only if we consider the modulation of a single wavepacket with a well-defined group velocity.

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Various perturbation methods, for instance the Krylov–Bogolubov–Mitropolsky (KBM) method as applied by Kakutani and Sugimoto [1], the reductive perturbation method developed by Taniuti and his collaborators [2, 3] and the asymptotic perturbation method [4–8] can be used to derive the NLS equation.

In particular, Calogero and Eckhaus [4–6] have demonstrated that model equations of NLS type are obtained from the class of nonlinear evolution equations with dispersive linear part and analytic nonlinear part. The starting hypothesis of a solution that is close to a solution of the linear part of the equation representing a single dispersive wave and is small, so the nonlinear effects are weak, leads to universal model equations of NLS type.

The interaction, and eventually the collisions, among solitary waves with different velocities that are not close to each other cannot be described by the above-mentioned methods and a different asymptotic reduction method [9, 10] based on a different spatio-temporal rescaling

$$\xi = \varepsilon^2 x \qquad \tau = \varepsilon^2 t \tag{2a}$$

must be used. The reduction method focuses on a solution that is small, due to the weak nonlinearity, and is close to a superposition of N dispersive waves ($N > 1$), with different group velocities.

In a previous paper [11], the particular case of ion acoustic waves in an unmagnetized plasma has been studied and a nonlinear partial differential system of equations describing N-interacting waves ($N > 1$) has been deduced for modulated amplitudes $\Psi_j = \Psi_j(\xi, \tau)$,

$$\Psi_{j,\tau} + V_j \Psi_{j,\xi} = \mathrm{i} a_j \Psi_0 \Psi_j + \mathrm{i} b_j \Phi_0 \Psi_j + \mathrm{i} \sum_{l=1}^{N} c_{jl} |\Psi_l|^2 \Psi_j \tag{2b}$$

$$\Psi_{0,\tau\tau} - \Psi_{0,\xi\xi} = \sum_{l=1}^{N} f_l (|\Psi_l|^2)_{\xi\xi} \tag{2c}$$

$$\Phi_{0,\tau\tau} - \Phi_{0,\xi\xi} = \sum_{l=1}^{N} g_l (|\Psi_l|^2)_{\xi\xi} \tag{2d}$$

where subscripts denote partial differentiation, a_j, b_j, c_{jl}, f_j, g_j are constant coefficients depending on wavenumber K_j and $V_j = V_j(K_j)$ is the relative group velocity. This system is C-integrable, i.e. can be linearized through an appropriate transformation of the dependent and independent variables. Localized solutions exist and give rise to envelope solitons. Each envelope soliton propagates with its own group velocity and during a collision maintains its shape, because a phase shift is the only change. Analytical predictions are supported by numerical results. We stress that this method is valid only if $N > 1$, because for $N = 1$ the NLS equation is recovered, either by the asymptotic perturbation method with the spatio-temporal rescaling (1*a*) or by the multiple scale methods.

In this paper the reduction method will be extended to the study of magnetized plasma. Also, in this case, envelope soliton dynamics leads to the NLS equation in the case of collisions among solitons with velocities that are close to each other [12, 13]. We consider a (2 + 1)-dimensional case and use the spatio-temporal rescaling

$$\xi = \varepsilon^2 x \qquad \eta = \varepsilon^2 y \qquad \tau = \varepsilon^2 t. \tag{3a}$$

We describe the slow modulation of amplitudes of N non-resonant waves. In the linear limit the solution is

$$\sum_{j=1}^{N} A_j \exp(\mathrm{i} z_j) \qquad z_j = K_{1,j} x + K_{2,j} y - \omega_j t \qquad j = 1, \ldots, N, \quad N > 1 \tag{3b}$$

where A_j and $\boldsymbol{K_j} \equiv (K_{1,j}, K_{2,j})$ are the complex amplitudes and the wavevectors, respectively. The frequency ω_j is obviously furnished by the relative dispersion relation $\omega_j = \omega_j(K_{1,j}, K_{2,j})$. The amplitudes of these dispersive waves (constant in the linear limit) are slowly modulated.

In section 2 we derive a model equation which describes the slow modulation and appears as a higher-dimensional generalization of the system (2*a*)–(2*c*). Subsequently, in section 3 we show that the obtained model system of equations is C-integrable. The Cauchy problem is resolved, just by quadratures, and explicit nontrivial solutions are constructed. We demonstrate the existence of solitons, i.e. coherent structures which preserve their shape during collisions, the only change being a phase shift.

The conclusion and final considerations are reserved for the last section.

2. Derivation of the model system

We consider ion acoustic waves in a two component electron collision dominated plasma. Electrons and ions have equal density. Moreover, electron inertia and ion temperature are neglected $(m_e/m_i \to 0, T_i/T_e \to 0)$. In the presence of an external magnetic field $\boldsymbol{B_0} = B_0\boldsymbol{x}$, the basic equations are

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\boldsymbol{v}) = 0 \tag{4a}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\phi + \Omega\boldsymbol{x}\wedge\boldsymbol{v} = 0 \tag{4b}$$

$$\nabla^2\phi = e^{\phi} - n \tag{4c}$$

where n is the ion density, ϕ is the electric field potential defined by $\boldsymbol{E} = -\nabla\phi$ and $\boldsymbol{v} \equiv (V_1, V_2, V_3)$ is the ion flow velocity. All these quantities are dimensionless, by means of the introduction of a characteristic length, the Debye length, $(K_BT_e/4\pi n_0e^2)^{\frac{1}{2}}$, a characteristic frequency, the ion plasma frequency, $(4\pi n_0e^2/m)^{\frac{1}{2}}$ and a characteristic electric field potential $\frac{K_BT_e}{e}$, where K_B is the Boltzmann constant and m is the ion mass (see for example [14]). Ω is a normalized measure of the strength of the magnetic field

$$\Omega = \frac{B_0}{c\sqrt{4\pi n_i m}} = \frac{V_A}{c} \tag{5}$$

where the velocity V_A is known as the Alfvén velocity. In this paper we study ion acoustic waves in very strong external magnetic fields and then $\Omega \approx 1$.

Note that the ion velocity is not curl-free and no velocity potential can be introduced.

After the replacement $n \to 1 + n$, we get

$$n_t + V_{1,x} + V_{2,y} + (nV_1)_x + (nV_2)_y = 0 \tag{6a}$$

$$\nabla^2\phi = e^{\phi} - 1 - n \tag{6b}$$

$$V_{1,t} + V_1V_{1,x} + V_2V_{1,y} + \phi_x = 0 \tag{6c}$$

$$V_{2,t} + V_1V_{2,x} + V_2V_{2,y} + \phi_y - \Omega V_3 = 0 \tag{6d}$$

$$V_{3,t} + V_1V_{3,x} + V_2V_{3,y} + \phi_z + \Omega V_2 = 0 \tag{6e}$$

where the subscripts denote partial differentiation. We have supposed variables to be z independent without loss of generality and then we have reduced perpendicular dependence to one coordinate [15].

First of all, we consider the linearized equation, i.e. the equation obtained after neglecting all the nonlinear terms:

$$n_t + V_{1,x} + V_{2,y} = 0 \tag{7a}$$
$$\nabla^2\phi - \phi + n = 0 \tag{7b}$$
$$V_{1,t} + \phi_x = 0 \tag{7c}$$
$$V_{2,t} + \phi_y - \Omega V_3 = 0 \tag{7d}$$
$$V_{3,t} + \phi_z + \Omega V_2 = 0. \tag{7e}$$

Fourier modes with constant amplitudes satisfy equations (7*a*)–(7*e*)

$$n \approx A_j \exp \mathrm{i}(K_{1,j}x + K_{2,j}y - \omega_j t) \tag{8a}$$
$$\phi \approx B_j \exp \mathrm{i}(K_{1,j}x + K_{2,j}y - \omega_j t) \tag{8b}$$
$$V_1 \approx C_j \exp \mathrm{i}(K_{1,j}x + K_{2,j}y - \omega_j t) \tag{8c}$$
$$V_2 \approx D_j \exp \mathrm{i}(K_{1,j}x + K_{2,j}y - \omega_j t) \tag{8d}$$
$$V_3 \approx E_j \exp \mathrm{i}(K_{1,j}x + K_{2,j}y - \omega_j t) \tag{8e}$$

if the following dispersion relation is verified

$$\omega_j^4(1 + K_j^2) - \omega_j^2(K_j^2 + \Omega^2(1 + K_j^2)) + \Omega^2 K_{1,j}^2 = 0. \tag{9}$$

Explicit expressions for the constant amplitudes are given in the appendix.

The group velocity $\boldsymbol{U}_j = (U_{1,j}, U_{2,j})$ (the speed with which a wavepacket peaked at that Fourier mode would move) is

$$U_{1,j} = \frac{\mathrm{d}\omega_j}{\mathrm{d}K_{1,j}} = \frac{K_{1,j}}{\omega_j}\frac{\omega_j^2(1+\Omega^2) - \omega_j^4 - \Omega^2}{2\omega_j^2(1+K_j^2) - (K_j^2 + \Omega^2(1+K_j^2))} \tag{10a}$$
$$U_{2,j} = \frac{\mathrm{d}\omega_j}{\mathrm{d}K_{2,j}} = \frac{K_{2,j}}{\omega_j}\frac{\omega_j^2(1+\Omega^2) - \omega_j^4}{2\omega_j^2(1+K_j^2) - (K_j^2 + \Omega^2(1+K_j^2))}. \tag{10b}$$

We assume that all the group velocities are different and not close to each other and consider a superposition of N dispersive waves, characterized by different values of the wavevector $\boldsymbol{K_j}$. We want to identify solutions of the nonlinear equations that are small of order ε and that are close in the limit of small ε to the solution (8*a*)–(8*e*). Weak nonlinearity induces a slow variation of the amplitudes of these dispersive waves and our aim is to obtain the nonlinear equations that describe such evolution, obviously in appropriate 'slow' and 'coarse-grained' variables defined by equation (1*c*). Since the amplitude of Fourier modes are not constant, higher order harmonics appear and in order to construct an approximate solution of the nonlinear equations (6*a*)–(6*e*) we introduce the asymptotic Fourier expansion

$$n(x, y, t) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(\mathrm{i}\sum_{j=1}^{N} n_j z_j\right)\varepsilon^{\gamma_{\underline{n}}}\psi_{\underline{n}}(\xi, \eta, \tau, \varepsilon) \tag{11a}$$
$$\phi(x, y, t) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(\mathrm{i}\sum_{j=1}^{N} n_j z_j\right)\varepsilon^{\gamma_{\underline{n}}}\lambda_{\underline{n}}(\xi, \eta, \tau, \varepsilon) \tag{11b}$$
$$V_1(x, y, t) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(\mathrm{i}\sum_{j=1}^{N} n_j z_j\right)\varepsilon^{\gamma_{\underline{n}}}\varphi_{\underline{n}}(\xi, \eta, \tau, \varepsilon) \tag{11c}$$
$$V_2(x, y, t) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(\mathrm{i}\sum_{j=1}^{N} n_j z_j\right)\varepsilon^{\gamma_{\underline{n}}}\mu_{\underline{n}}(\xi, \eta, \tau, \varepsilon) \tag{11d}$$

$$V_3(x,y,t) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(\mathrm{i}\sum_{j=1}^{N} n_j z_j\right)\varepsilon^{\tilde{\gamma}_{\underline{n}}}\upsilon_{\underline{n}}(\xi,\eta,\tau,\varepsilon) \tag{11e}$$

where the index $\underline{n}$ stands for the set $\{n_j;\ j = 1, 2, \ldots N\}$. The functions $\varphi_{\underline{n}}(\xi,\eta,\tau,\varepsilon)$, $\psi_{\underline{n}}(\xi,\eta,\tau,\varepsilon)$, $\lambda_{\underline{n}}(\xi,\eta,\tau,\varepsilon)$, $\mu_{\underline{n}}(\xi,\eta,\tau,\varepsilon)$ and $\upsilon_{\underline{n}}(\xi,\eta,\tau,\varepsilon)$ depend parametrically on ε and we assume that their limit for $\varepsilon \to 0$ exists and is finite. We moreover assume that the following conditions hold:

$$\gamma_{\underline{n}} = \gamma_{-\underline{n}} \qquad \gamma_{\underline{n}} = 2 \qquad \tilde{\gamma}_{\underline{n}} = 3 \qquad \text{if} \quad n_j = 0, \qquad \text{for} \quad j = 1, 2, \ldots, N \tag{12a}$$

$$\tilde{\gamma}_{\underline{n}} = \gamma_{\underline{n}} = \sum_{j=1}^{N} |n_j| \qquad \text{otherwise.} \tag{12b}$$

This implies that we obtain the main amplitudes if one of the indices n_j has unit modulus and all the others vanish. We use the following notations

$$\psi_{\underline{n}} = \Psi_j \qquad \text{if} \quad n_j = 1 \quad \text{and} \quad n_m = 0 \qquad \text{for} \quad j \neq m \tag{13a}$$

$$\psi_{\underline{n}} = \Psi_0 \qquad \text{if} \quad n_j = 0 \qquad \text{for} \quad j = 1, 2, \ldots, N \tag{13b}$$

$$\psi_{\underline{n}} = \Psi_{2,j} \qquad \text{if} \quad n_j = 2 \quad \text{and} \quad n_m = 0 \qquad \text{for} \quad j \neq m \tag{13c}$$

$$\psi_{\underline{n}} = \Psi_{11,jm} \qquad \text{if} \quad n_j = n_m = 1 \quad \text{and} \quad n_l = 0 \qquad \text{for} \quad l \neq j, m \tag{13d}$$

$$\psi_{\underline{n}} = \Psi_{1-1,jm} \qquad \text{if} \quad n_j = 1, \quad n_m = -1 \quad \text{and} \quad n_l = 0 \qquad \text{for} \quad l \neq j, m, \quad j \neq m. \tag{13e}$$

Similar notations are employed for equations (11*b*)–(11*e*). For example $\lambda_{\underline{n}} = L_{11,jm}$, if $n_j = n_m = 1$ and $n_l = 0$ for $l \neq j, m$, or $\varphi_{\underline{n}} = \Phi_j$, if $n_j = 1$ and $n_m = 0$ for $j \neq m$ and so on. Taking into account (12*a*), (12*b*) and (13*a*)–(13*e*), equation (11*a*) can be written more explicitly in the following form

$$\begin{aligned} u(x,y,t) = {} & \varepsilon \sum_{j=1}^{N} [\exp(\mathrm{i}z_j)\Psi_j + \varepsilon \exp(2\mathrm{i}z_j)\Psi_{2,j} + \text{c.c.}] + \varepsilon^2\Psi_0 \\ & + \varepsilon^2 \sum_{\substack{j,m=1 \\ (j\neq m)}}^{N} [\exp(\mathrm{i}(z_j + z_m)\Psi_{1+1,jm} + \text{c.c.}] \\ & + \varepsilon^2 \sum_{\substack{j,m=1 \\ (j\neq m)}}^{N} [\exp(\mathrm{i}(z_j - z_m)\Psi_{1-1,jm} + \text{c.c.}] + \mathrm{o}(\varepsilon^3) \end{aligned} \tag{14}$$

where c.c. stands for complex conjugate. Also, in this case similar expressions are valid for equations (11*b*)–(11*e*). Note that the variable change (1*c*) implies that differentiation with respect to the fast variables x and t must be substituted in the following way

$$\partial_t \to \varepsilon^2\partial_\tau - \mathrm{i}\sum_{j=1}^{N} n_j\omega_j \qquad \partial_x \to \varepsilon^2\partial_\xi + \mathrm{i}\sum_{j=1}^{N} n_j K_{1,j} \qquad \partial_y \to \varepsilon^2\partial_\eta + \mathrm{i}\sum_{j=1}^{N} n_j K_{2,j}. \tag{15}$$

Substituting (11*a*)–(11*e*) in equations (6*a*)–(6*e*) and considering the different equations obtained for every harmonic and for a fixed order of approximation in ε, we obtain for $n_j = 2$, $n_m = 0$, if $m \neq j$,

$$\Psi_{2,j} = B_{1,j}\Psi_j^2 + \text{h.o.t.} \qquad \Phi_{2,j} = B_{2,j}\Psi_j^2 + \text{h.o.t.} \tag{16a}$$

$$L_{2,j} = B_{3,j}\Psi_j^2 + \text{h.o.t.} \qquad M_{2,j} = B_{4,j}\Psi_j^2 + \text{h.o.t.} \qquad N_{2,j} = B_{5,j}\Psi_j^2 + \text{h.o.t.} \tag{16b}$$

(*a*)

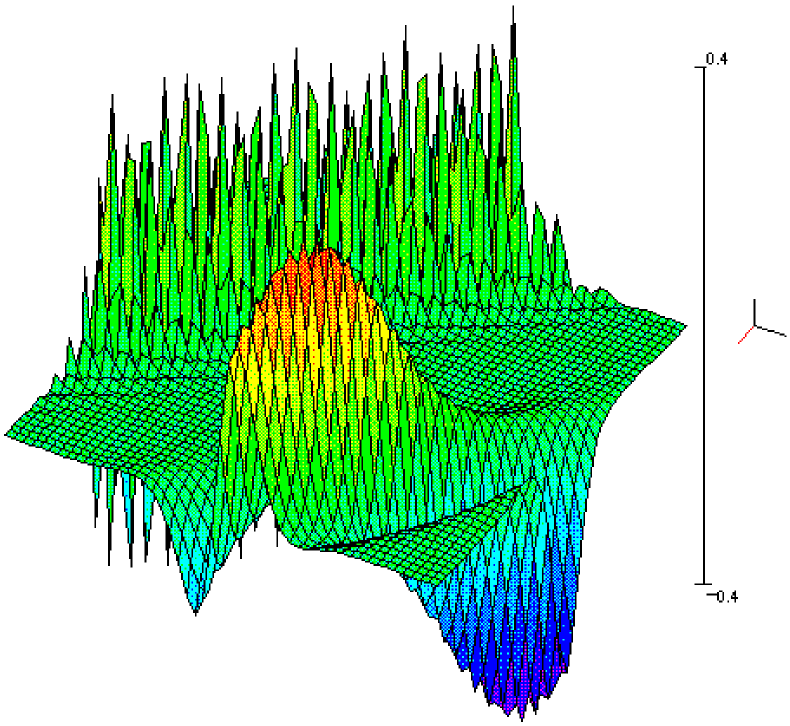

Figure 1. Two solitons with the same amplitude but different wavenumbers (with $A_1 = A_2 = 0.1$, $K_{1,1} = 1.0$, $K_{2,1} = 0.6$, $K_{1,2} = -0.1$, $K_{2,2} = -0.2$). (*a*) Initial condition; (*b*) undergoing a collision and; (*c*) separation. We can see that solitons preserve their shapes but with a phase shift.

where we have used the notation (13*c*) and h.o.t. = higher order terms. Explicit expressions of the constant coefficients $B_{1,j}$, $B_{2,j}$, $B_{3,j}$, $B_{4,j}$, $B_{5,j}$ are given in the appendix.

In a similar way for $n_j = n_m = 1$, $n_l = 0$, if $l \neq j, m$, we get to the order of ε^2:

$$\Psi_{11,jm} = C_{1,jm}\Psi_j\Psi_m,\ \Phi_{11,jm} = C_{2,jm}\Psi_j\Psi_m,\ L_{11,jm} = C_{3,jm}\Psi_j\Psi_m \tag{17a}$$

$$M_{11,jm} = C_{4,jm}\Psi_j\Psi_m,\ N_{11,jm} = C_{5,jm}\Psi_j\Psi_m \tag{17b}$$

where we have used the notation (13*d*) (see the appendix for the explicit expression of coefficients). For $n_j = 1$, $n_m = -1$, $n_l = 0$, if $l \neq j, m$:

$$\Psi_{1-1,jm} = D_{1,jm}\Psi_j\Psi_m^*,\ \Phi_{1-1,jm} = D_{2,jm}\Psi_j\Psi_m^*,\ L_{1-1,jm} = D_{3,jm}\Psi_j\Psi_m^* \tag{18a}$$

$$M_{1-1,jm} = D_{4,jm}\Psi_j\Psi_m^*,\ N_{1-1,jm} = D_{5,jm}\Psi_j\Psi_m^* \tag{18b}$$

where we have used the notation (13*e*) (see the appendix).

Equation (6*a*) for $n_j = 1$, $n_m = 0$, if $j \neq m$ gives to the order of ε^3:

$$\Psi_{j,\tau} + \mathrm{i}K_{1,j}\tilde{\Phi}_j + \mathrm{i}K_{2,j}\tilde{M}_j + \Phi_{j,\xi} + M_{j,\xi} + \mathrm{i}K_{1,j}\Bigg(\Psi_0\Phi_j + \Psi_j\Phi_0 + \Psi_{2,j}\Phi_j^* + \Psi_j^*\Phi_{2,j} + \sum_{m=1(m\neq j)}^{N}(\Psi_{11,jm}\Phi_m^* + \Psi_{1-1,jm}\Phi_m + \Phi_{11,jm}\Psi_m^* + \Phi_{1-1,jm}\Psi_m)\Bigg)$$

(*b*)

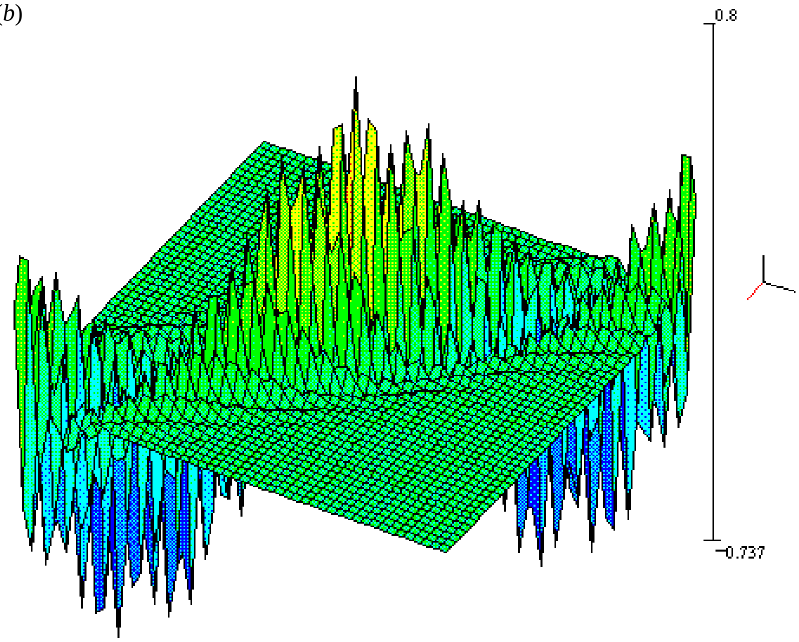

Figure 1. (Continued)

$$+\mathrm{i}K_{2,j}\Bigg(\Psi_0 M_j + \Psi_j M_0 + \Psi_{2,j} M_j^* + \Psi_j^* M_{2,j} + \sum_{m=1(m\neq j)} (\Psi_{11,jm} M_m^* + \Psi_{1-1,jm} M_m + M_{11,jm}\Psi_m^* + M_{1-1,jm}\Psi_m)\Bigg) = 0 \qquad (19)$$

where $\tilde{\Phi}_j$, $\tilde{M}_j$ are the correction of order ε^3 to Φ_j and M_j. These corrections can be evaluated by considering equations (6*b*)–(6*e*) for $n_j = 1$, $n_m = 0$, if $j \neq m$ and substituted into equation (19). After long calculations, we arrive at a system of equations for the N modulated amplitudes $\Psi_j(\xi, \eta, \tau)$,

$$\Psi_{j,\tau} + U_{1,j}\Psi_{j,\xi} + U_{2,j}\Psi_{j,\eta} = \mathrm{i}a_j\Psi_0\Psi_j + \mathrm{i}b_j\Phi_0\Psi_j + \mathrm{i}\sum_{l=1}^{N} c_{jl}|\Psi_l|^2\Psi_j \qquad (20a)$$

where a_j, b_j, c_{jl} are constant coefficients depending on the wavenumber $\boldsymbol{K_j} \equiv (K_{1,j}, K_{2,j})$ (their explicit form is given in the appendix).

From equations (6*a*)–(6*e*) for $n_j = 0$ to the order of ε^4 we find

$$\Psi_{0,\tau\tau} - \Psi_{0,\xi\xi} = \sum_{l=1}^{N}(f_l(|\Psi_l|^2)_{\xi\xi} + g_l(|\Psi_l|^2)_{\xi\eta} + h_l|\Psi_l|^2_{\eta\eta}) \qquad (20b)$$

(*c*)

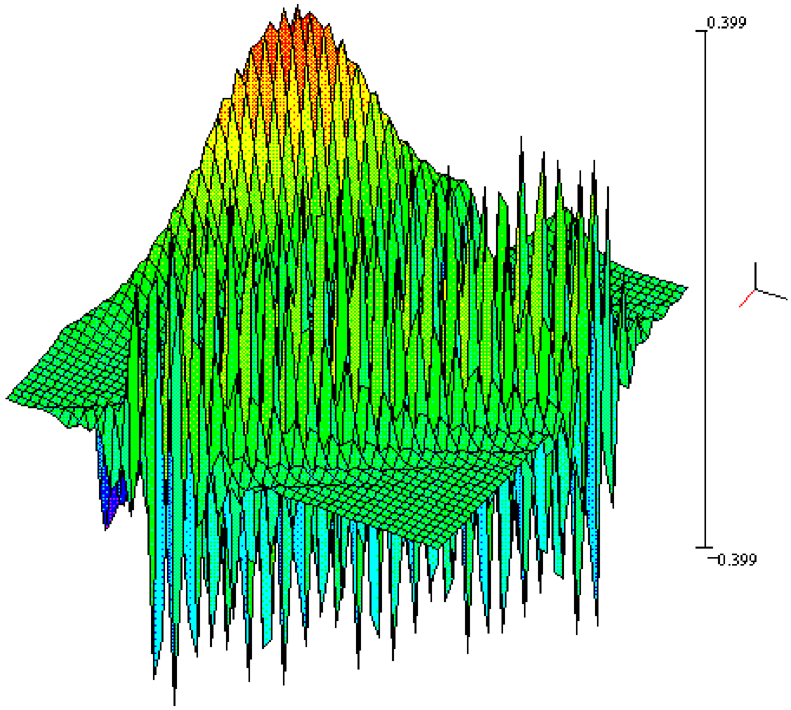

Figure 1. (Continued)

$$\Phi_{0,\tau\tau} - \Phi_{0,\xi\xi} = \sum_{l=1}^{N} (\tilde{f}_l(|\Psi_l|^2)_{\xi\xi} + \tilde{g}_l(|\Psi_l|^2)_{\xi\eta}) \tag{20c}$$

where f_j, g_j, h_j, $\tilde{f}_j$, $\tilde{g}_j$, are constant coefficients depending on the wavenumber K_j (their explicit form is given in the appendix).

The validity of the approximate solution should be expected to be restricted on bounded intervals of the τ-variable and on timescale $t = \mathrm{O}(\frac{1}{\varepsilon^2})$. If one wishes to study solutions on intervals such that $\tau = \mathrm{O}(\frac{1}{\varepsilon})$ then the higher terms will in general affect the solution and must be included.

A comparison between the starting equations (6*a*)–(6*e*) and the final system (20*a*)–(20*c*) shows that the slow and coarse-grained character of the independent variables leads to a simpler system of equations, because many specific details characterizing the original equations get smoothed away.

In the next section we will demonstrate that the system of equations (20*a*)–(20*c*) is integrable by means of an appropriate transformation of the dependent and the independent variables (C-integrability).

As we can see from equations (20*a*)–(20*c*), the system obtained does not reduce to the NLS equation for $N = 1$ (one single wave), because we have used the rescaling (1*c*). If we use the

(*a*)

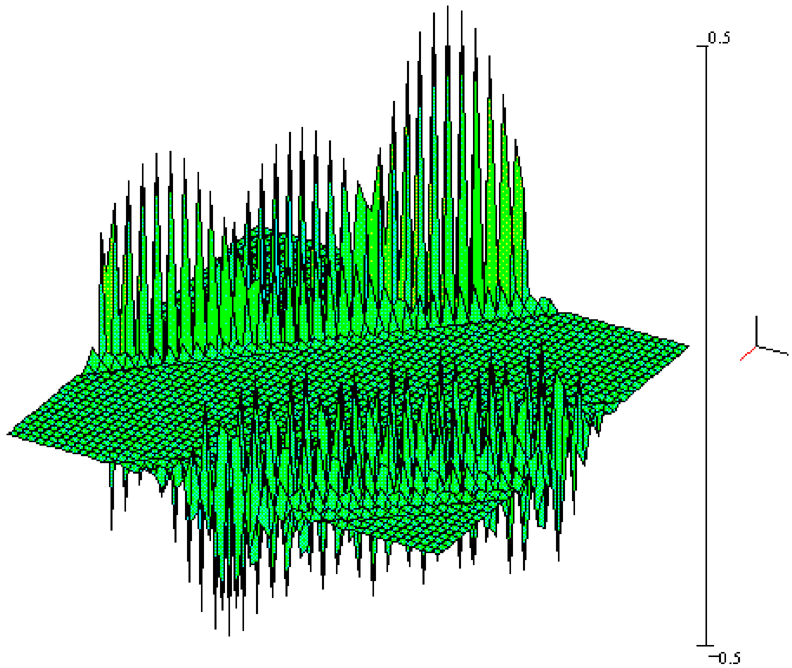

Figure 2. A collision between two solitons with different amplitudes ($A_1 = 0.25$, $A_2 = 0.1$, $K_{1,1} = 1.1$, $K_{2,1} = 1.0$, $K_{1,2} = -1.2$, $K_{2,2} = -0.9$). (*a*) Initial condition; (*b*) the collision and (*c*) the final outcome.

spatio-temporal rescaling (1) for $N = 1$, we derive the NLS equation to describe modulation of the amplitude [4–6]. Moreover, a system of equations of NLS type can be obtained only if the group velocities are equal or appropriately close to each other [8].

3. Integrability of the model system of equations

In this section we demonstrate that the system of nonlinear equations (20*a*)–(20*c*) is C-integrable. We set

$$\Psi_j(\xi, \eta, \tau) = \rho_j(\xi, \eta, \tau) \exp[\mathrm{i}\vartheta_j(\xi, \eta, \tau)] \qquad j = 1, \ldots, N \tag{21}$$

with $\rho_j = \rho_j(\xi, \eta, \tau) > 0$ and $\theta_j = \vartheta_j(\xi, \eta, \tau)$ real functions. Then equation (20*a*) yields

$$\rho_{j,\tau}(\xi, \eta, \tau) + U_{1,j}\rho_{j,\xi}(\xi, \eta, \tau) + U_{2,j}\rho_{j,\eta}(\xi, \eta, \tau) = 0 \tag{22}$$

$$\theta_{j,\tau}(\xi, \eta, \tau) + U_{1,j}\vartheta_{j,\xi}(\xi, \eta, \tau) + U_{2,j}\vartheta_{j,\eta}(\xi, \eta, \tau) = a_j\Psi_0(\xi, \eta, \tau) + b_j\Phi_0(\xi, \eta, \tau) + \sum_{l=1}^{N} c_{jl}|\rho_l(\xi, \eta, \tau)|^2. \tag{23}$$

The general solution for the Cauchy problem of (22) reads

$$\rho_j(\xi, \eta, \tau) = \rho_j(\beta_j\tau + \alpha_{1,j}\xi + \alpha_{2,j}\eta) \tag{24a}$$

(*b*)

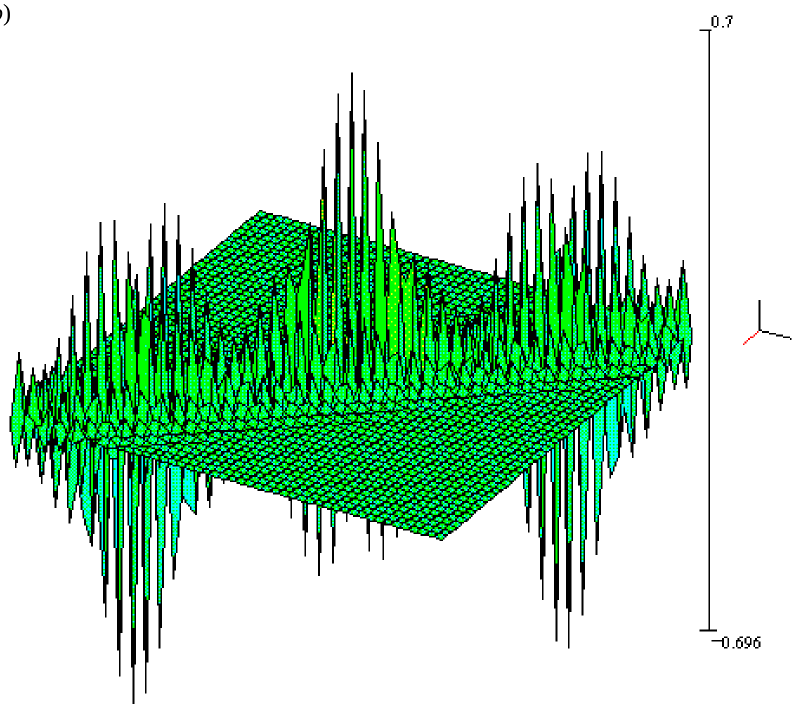

Figure 2. (Continued)

where the N real functions $\rho_j(\alpha_{1,j}\xi + \alpha_{2,j}\eta)$, which represent the initial shape, can be chosen arbitrarily and $\beta_j, \alpha_{1,j}, \alpha_{2,j}$ are real constants which satisfy the relation

$$\beta_j + \alpha_{1,j}U_{1,j} + \alpha_{2,j}U_{2,j} = 0. \tag{24b}$$

The general solution for (20*b*) and (20*c*) is

$$\Psi_0(\xi, \eta, \tau) = \gamma_1(\xi - \tau, \eta) + \gamma_2(\xi + \tau, \eta) + \sum_{j=1}^{N} \frac{f_j\alpha_{1,j}^2 + g_j\alpha_{1,j}\alpha_{2,j} + h_j\alpha_{2,j}^2}{\beta_j^2 - \alpha_{1,j}^2}\{[\rho_j(\beta_j\tau + \alpha_{1,j}\xi + \alpha_{2,j}\eta)]^2\} \tag{25}$$

$$\Phi_0(\xi, \eta, \tau) = \gamma_3(\xi - \tau, \eta) + \gamma_4(\xi + \tau, \eta) + \sum_{j=1}^{N} \frac{\tilde{f}_l\alpha_{1,j}^2 + \tilde{g}_j\alpha_{1,j}\alpha_{2,j}}{\beta_j^2 - \alpha_{1,j}^2}\{[\rho_j(\beta_j\tau + \alpha_{1,j}\xi + \alpha_{2,j}\eta)]^2\}. \tag{26}$$

Here the four real functions $\gamma_1(\xi, \eta), \gamma_2(\xi, \eta), \gamma_3(\xi, \eta), \gamma_4(\xi, \eta)$ can be chosen arbitrarily and their shapes are determined by the initial data.

The general solution of (23) is

$$\theta_j(\xi, \eta, \tau) = \delta_j(\tilde{\beta}_j\tau + \tilde{\alpha}_{1,j}\xi + \tilde{\alpha}_{2,j}\eta) + a_j\int_0^\tau \Psi_0(\xi - U_{1,j}(\tau - \tilde{\tau}), \eta - U_{2,j}(\tau - \tilde{\tau}), \tilde{\tau})\,\mathrm{d}\tilde{\tau}$$
$$+b_j\int_0^\tau \Phi_0(\xi - U_{1,j}(\tau - \tilde{\tau}), \eta - U_{2,j}(\tau - \tilde{\tau}), \tilde{\tau})\,\mathrm{d}\tilde{\tau}$$

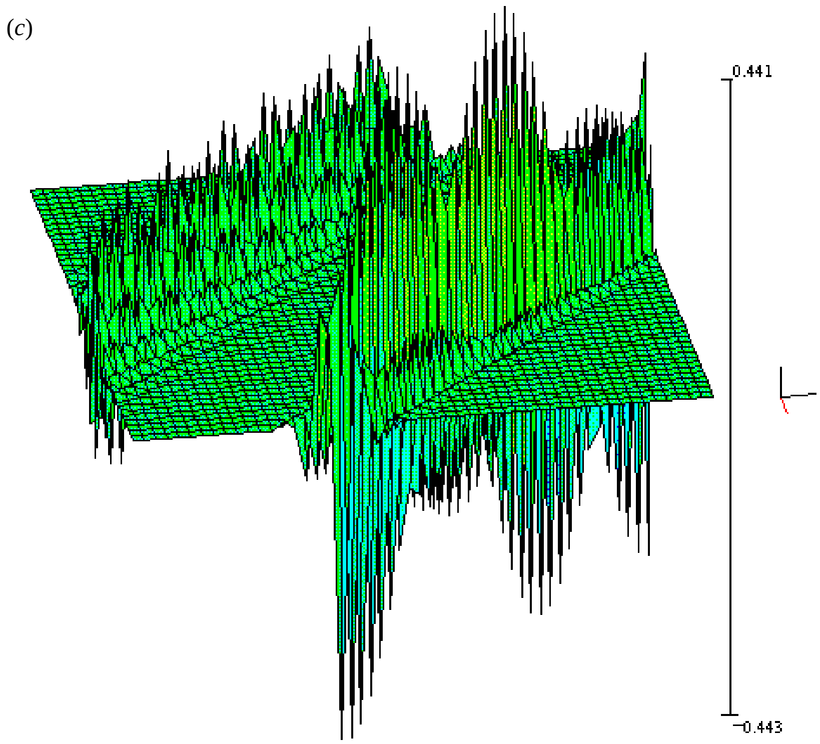

Figure 2. (Continued)

$$+\sum_{l=1}^{N} c_{jl} \int_0^{\tau} (\rho_l(\xi - U_{1,j}(\tau - \tilde{\tau}), \eta - U_{2,j}(\tau - \tilde{\tau}), \tilde{\tau}))^2 \, \mathrm{d}\tilde{\tau} \tag{27}$$

where the N functions $\delta_j(\alpha_{1,j}\xi + \alpha_{2,j}\eta)$ are fixed by the initial data.

The approximate solution good to the order of ε for the system of equations (6*a*)–(6*e*) is

$$n(x, y, t) = 2\varepsilon \sum_{j=1}^{N} \rho_j \cos(\vartheta_j + K_{1,j}x + K_{2,j}y - \omega_j t) + \mathrm{o}(\varepsilon^2) \tag{28a}$$

$$\phi(x, y, t) = 2\varepsilon \sum_{j=1}^{N} \rho_j B_j \cos(\vartheta_j + K_{1,j}x + K_{2,j}y - \omega_j t) + \mathrm{o}(\varepsilon^2) \tag{28b}$$

$$V_1(x, y, t) = 2\varepsilon \sum_{j=1}^{N} \rho_j C_j \cos(\vartheta_j + K_{1,j}x + K_{2,j}y - \omega_j t) + \mathrm{o}(\varepsilon^2) \tag{28c}$$

$$V_2(x, y, t) = 2\varepsilon \sum_{j=1}^{N} \rho_j D_j \cos(\vartheta_j + K_{1,j}x + K_{2,j}y - \omega_j t) + \mathrm{o}(\varepsilon^2) \tag{28d}$$

$$V_3(x, y, t) = 2\varepsilon \sum_{j=1}^{N} \rho_j E_j \cos(\vartheta_j + K_{1,j}x + K_{2,j}y - \omega_j t) + \mathrm{o}(\varepsilon^2). \tag{28e}$$

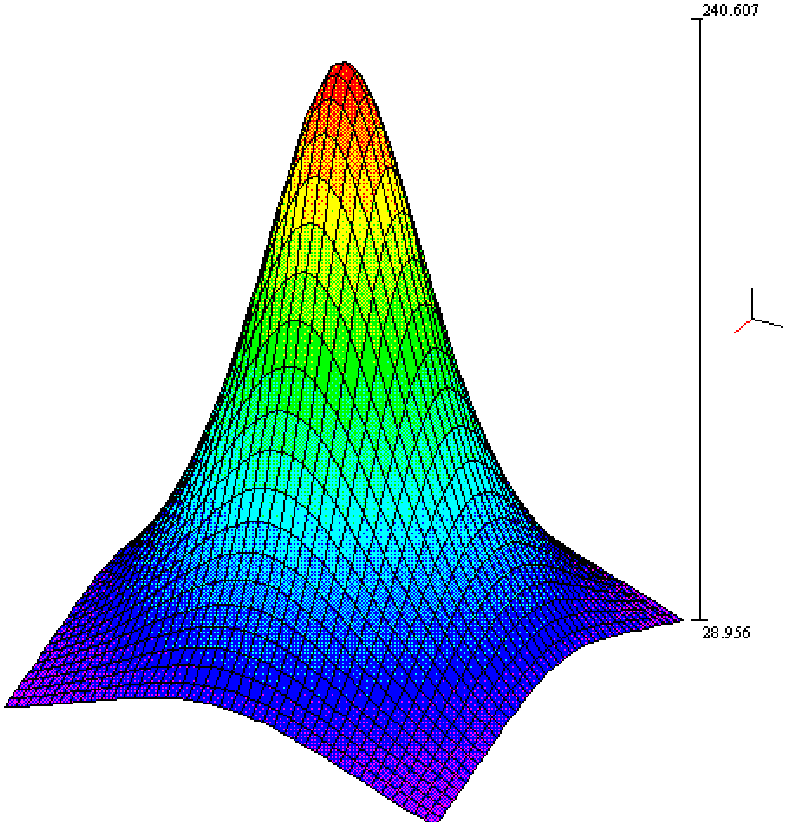

Figure 3. Dispersion relation ω as function of K_1 and K_2.

The corrections of order ε^2 to the approximate solution depend on higher harmonics and can be easily calculated from results obtained in section 2 and in the appendix.

The simplest solution of the system (20*a*)–(20*c*) is the plane wave

$$\Psi_0 = \Phi_0 = 0 \qquad \rho_j = A_j = \text{constant} \qquad \theta_j = \tilde{K}_1\xi + \tilde{K}_2\eta - \tilde{\omega}\tau \tag{29}$$

where the amplitudes and phases are connected according to the dispersion relation

$$\tilde{\omega}_j = U_{1,j}\tilde{K}_{1,j} + U_{2,j}\tilde{K}_{2,j} - \sum_{m=1}^{N} c_{jm}A_m^2. \tag{30}$$

Thanks to the C-integrable nature of the system (20*a*)–(*c*) more interesting characteristics can be exhibited explicitly. It is possible for there to be a group of N solitons which interact with each other, preserving their shapes, and that propagate with the relative group velocity $\boldsymbol{U_j} = (U_{1,j}, U_{2,j})$ (see (10)). These N solitons have fixed speeds but arbitrary shapes.

The collision of two solitons does not produce a change in the amplitude ρ_j of each of them, but only a change in the phase given by equation (27). Every solution, in the remote past and future, factors into N separate solitons. The soliton order can only determine the sequence of pair collisions or the eventual occurrence of multiple collisions but we stress that this final outcome is independent of the localization of the solitons in the remote past. This last feature justifies the use of the term soliton to describe these behaviours.

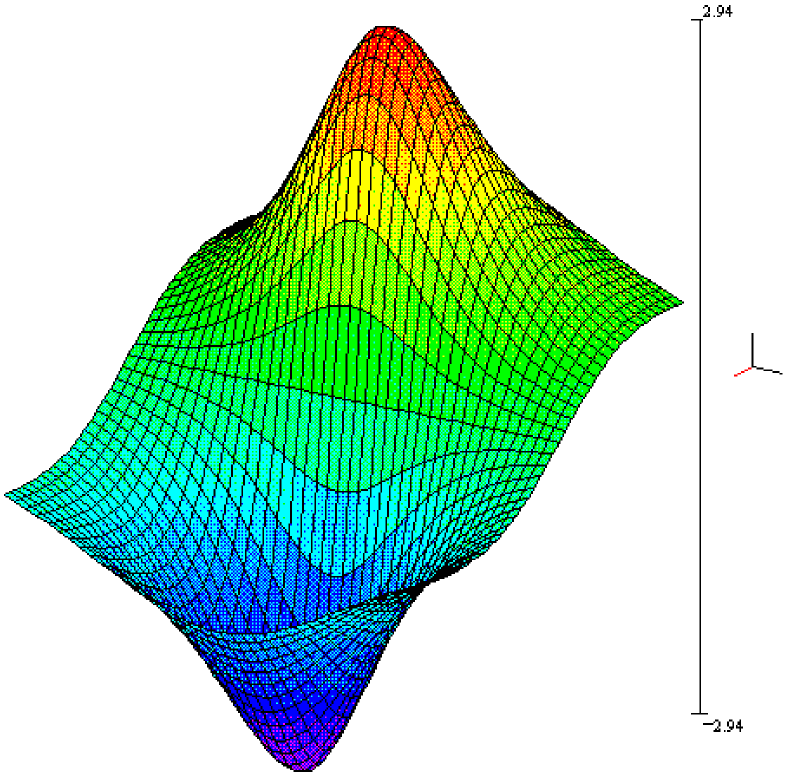

Figure 4. Group velocity U_1 as function of K_1 and K_2.

From (28) we deduce the presence of a shift in the amplitude oscillation. For instance, we take $N = 2$ and

$$\rho_j(\xi, \eta, \tau) = \frac{2A_j}{ch(2A_j(\alpha_{1,j}\xi + \alpha_{2,j}\eta + \beta_j\tau))} \tag{31a}$$

$$\gamma_i = 0 \qquad \text{for} \quad 1, \ldots, 4 \quad \text{and} \quad \delta_j = 0 \quad \text{for} \quad j = 1, \ldots, N \tag{31b}$$

then

$$\Psi_0(\xi, \eta, \tau) = \sum_{j=1}^{N} \frac{f_j\alpha_{1,j}^2 + g_j\alpha_{1,j}\alpha_{2,j} + h_j\alpha_{2,j}^2}{\beta_j^2 - \alpha_{1,j}^2} \left\{ \frac{4A_j^2}{ch^2(2A_j(\beta_j\tau + \alpha_{1,j}\xi + \alpha_{2,j}\eta))} \right\} \tag{32}$$

$$\Phi_0(\xi, \eta, \tau) = \sum_{j=1}^{N} \frac{\tilde{f}_j\alpha_{1,j}^2 + \tilde{g}_j\alpha_{1,j}\alpha_{2,j}}{\beta_j^2 - \alpha_{1,j}^2} \left\{ \frac{4A_j^2}{ch^2(2A_j(\beta_j\tau + \alpha_{1,j}\xi + \alpha_{2,j}\eta))} \right\} \tag{33}$$

$$\begin{aligned}\theta_j(\xi, \eta, \tau) = \sum_{l=1}^{N} &[4A3_l(c_{jl}(\beta_l^2 - \alpha_{1,l}^2) + b_j(\tilde{f}_l\alpha_{1,j}^2 + \tilde{g}_j\alpha_{1,j}\alpha_{2,j}) \\ &+ a_j(f_l\alpha_{1,l}^2 + g_l\alpha_{1,l}\alpha_{1,l}\alpha_{2,l} + h_l\alpha_{2,l}^2))](\beta_l^2 - \alpha_{1,l}^2) \\ &\times \frac{\tau}{ch^2(2A_l(\alpha_{1,j}(\xi - U_{1,j}\tau) + \alpha_{2,j}(\zeta - U_{2,j}\tau)))}\end{aligned} \tag{34}$$

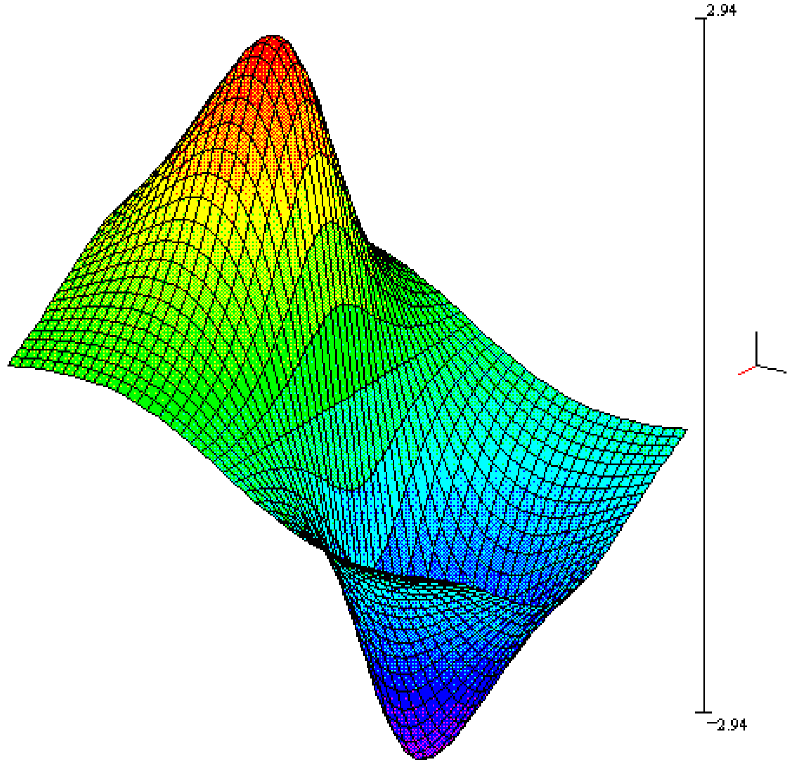

Figure 5. Group velocity U_2 as function of K_1 and K_2.

where A_j, $j = 1, \ldots, N$, are real constants of order ε. Substituting (31) and (34) into (28) we obtain the approximate solution good to the order of ε. Each solitary wave advances with a constant velocity (the group velocity) before and after collisions. Only the phase is changed during collisions owing to the presence of the other solitary waves. Results are illustrated in the following example. For the initial conditions we have chosen

$$n(x, y, 0) = 4\sum_{j=1}^{2} \frac{A_j}{ch(2A_j(\alpha_{1,j}(x-\bar{x}_j)+\alpha_{2,j}(y-\bar{y}_j)))}\cos(K_{1,j}x + K_{2,j}y) \tag{35}$$

where the initial positions are $\bar{x}_1 = -40$, $\bar{y}_1 = 0$, $\bar{x}_2 = 25$, $\bar{y}_2 = 25$.

The approximate solution good to the order of ε is given by

$$n(x, y, t) = 4\sum_{j=1}^{2} \frac{A_j \cos(K_{1,j}x + K_{2,j}y + \vartheta_j - \omega_j t)}{ch(2A_j(\alpha_{1,j}(x-\bar{x}_j)+\alpha_{2,j}(y-\bar{y}_j)+\beta_j t))} \tag{36}$$

where θ_j can be calculated from (34).

In figure 1 we show the collision of two solitons with the same amplitude ($A_1 = A_2 = 0.1$, $K_{1,1} = 1.0$, $K_{2,1} = 0.6$, $K_{1,2} = -0.1$, $K_{2,2} = -0.2$). The initial condition is shown in figure 1(*a*), then the two solitons collide (figure 1(*b*)) and then separate (figure 1(*c*)). We can see that solitons preserve their shapes but with a phase shift. In figure 2 we show a collision between two solitons with different amplitude ($A_1 = 0.25$, $A_2 = 0.1$, $K_{1,1} = 1.1$, $K_{2,1} = 1.0$, $K_{1,2} = -1.2$, $K_{2,2} = -0.9$).

4. Discussion of results and conclusion

Evolution of the envelopes of N non-resonant dispersive waves in a magnetized plasma has been considered in appropriately coarse-grained and slow variables. A system of equations describing the behaviour of ion magneto-acoustic waves as superpositions of plane waves, the amplitude of which is modulated by the nonlinear terms, has been derived by means of an asymptotic reduction method. For the solution we assume a Fourier expansion in which the coefficients are power series of a small parameter and vary slowly in space and time. Substituting the expression of the solution into the original equation and projecting onto each Fourier mode we have derived dynamic equations for the modulated amplitudes (see equations (20*a*)–(20*c*)).

This C-integrable system can describe solitons moving with different and not close to each other velocities. Resolving the Cauchy problem and choosing an appropriate nontrivial initial condition we demonstrate that solitons preserve their shapes during collisions and the only change is a modification in the phase of the amplitude oscillation.

In order to establish when the group velocity is notably different for different wavevectors, we show the dispersion relation $\omega = \omega(K_1, K_2)$ in figure 3 and the group velocity components $U_1 = U_1(K_1, K_2)$ and $U_2 = U_2(K_1, K_2)$ in figures 4 and 5 for $\Omega = 1$ (Ω is a normalized measure of the strength of the magnetic field, see equation (5)).

As we can see, the group velocity of Fourier modes is rapidly varying from negative to positive values of K_1 or K_2 and then the condition of different velocities can be easily satisfied. The shapes of these surfaces are independent of small variations of Ω.

Now we discuss briefly some possible developments of this type of research.

(i) Derivation of the model equation for other non-resonant plasma waves, for example the electron plasma waves.

(ii) Derivation of the model equation for the interactions among phase resonant waves, when the following conditions are simultaneously verified

$$\sum_{j=1}^{N} K_{1,j} = 0 \qquad \sum_{j=1}^{N} K_{2,j} = 0 \qquad \sum_{j=1}^{N} \omega_j = 0 \qquad (N > 2) \tag{37}$$

and we assume that none of the wavenumbers vanish and that they differ in modulus ($K_j \neq 0$ for $j = 1, 2, \ldots, N$ and $K_j \neq \pm K_l$ if $j \neq l$).

(iii) Study of the behaviour of the solutions, and in particular of the envelope soliton solutions, beyond the leading order in the expansion parameter ε.

Appendix

The linearized equations are satisfied by Fourier modes with amplitudes

$$B_j = \frac{A_j}{1 + K_j^2} \qquad C_j = \frac{A_j K_{1,j}}{\omega_j (1 + K_j^2)} \qquad D_j = \frac{A_j(\omega_j^2(1 + K_j^2) - K_{1,j}^2)}{K_{2,j}\omega_j(1 + K_j^2)} \tag{A.1a}$$

$$E_j = \frac{-\mathrm{i}A_j\Omega(\omega_j^2(1 + K_j^2) - K_{1,j}^2)}{K_{2,j}\omega_j^2(1 + K_j^2)}. \tag{A.1b}$$

For the coefficients in equations (16*a*) and (16*b*) we find

$$B_{1,j} = \frac{\omega_j(1 + 4K_j^2)(4\omega_j^2 - \Omega^2 - 4\Omega^2 K_j^2)}{(1 + 4K_j^2)(-4\omega_j^4 + 4\omega_j^2 K_{2,j}^2 - \Omega^2 K_{1,j}^2 + \Omega^2\omega_j^2 + 4\Omega^2\omega_j^2 K_j^2) + 4\omega_j^2 K_{1,j}^2}$$

$$\times\left(\frac{2\omega_j K_{2,j}}{4\omega_j^2-\Omega^2}(K_{1,j}C_jD_j+K_{2,j}D_j^2)-\frac{K_{2,j}B_j^2}{1+4K_j^2}\right.$$
$$+\frac{\mathrm{i}\Omega E_j}{2\omega_j}(K_{1,j}C_j+D_jK_{2,j})+\frac{K_{1,j}^2B_j^2}{2\omega_j(1+4K_j^2)}$$
$$\left.-\left(\frac{K_{1,j}C_j}{2\omega_j}+1\right)(K_{1,j}C_j+K_{2,j}D_j)\right) \quad \text{(A.2}a\text{)}$$

$$B_{2,j}=\frac{2B_{1,j}-B_j^2}{2(1+4K_j^2)} \qquad B_{3,j}=\frac{2K_{1,j}B_{2,j}+K_{1,j}C_j^2+K_{2,j}C_jD_j}{2\omega_j} \quad \text{(A.2}b\text{)}$$

$$B_{4,j}=\frac{2\omega_j(K_{1,j}C_jD_j+K_{2,j}D_j^2+2K_{2,j}B_{2,j})+\mathrm{i}\Omega E_j(K_{1,j}C_j+K_{2,j}D_j)}{4\omega_j^2-\Omega^2} \quad \text{(A.2}c\text{)}$$

$$B_{5,j}=\frac{E_j(C_jK_{1,j}+D_jK_{2,j})-\mathrm{i}\Omega B_{4,j}}{2\omega_j}. \quad \text{(A.2}d\text{)}$$

For the coefficients in equations (17*a*) and (17*b*) we obtain

$$C_{3,jm}=\frac{(2-B_jB_m)}{2(1+(K_{1,j}+K_{1,m})^2+(K_{2,j}+K_{2,m})^2)} \quad \text{(A.3}a\text{)}$$

$$C_{2,jm}=\frac{((K_{1,j}+K_{1,m})(C_{3,jm}+C_jC_m)+K_{2,m}D_jC_m+K_{2,j}D_mC_j)}{(\omega_j+\omega_m)} \quad \text{(A.3}b\text{)}$$

$$C_{4,jm}=\frac{1}{\Omega^2+(\omega_j+\omega_m)^2}(-\Omega(K_{2,j}D_mE_j+K_{2,m}D_jE_m+K_{1,j}C_mD_j+K_{1,m}C_jE_m)$$
$$+(\omega_j+\omega_m)(K_{1,j}C_mD_j+(K_{2,j}+K_{2,m})(D_jD_m+C_{3,jm})+K_{1,m}C_jD_m)) \quad \text{(A.3}c\text{)}$$

$$C_{5,jm}=\frac{(\Omega C_{4,jm}+K_{1,m}C_jE_m+K_{1,j}C_mE_j+K_{2,m}D_jE_m+K_{2,j}D_mE_j)}{(\omega_j+\omega_m)} \quad \text{(A.3}d\text{)}$$

$$C_{1,jm}=\frac{((K_{1,j}+K_{1,m})(C_m+C_j+C_{2,jm})+(K_{2,j}+K_{2,m})(C_{4,jm}+D_j+D_m))}{(\omega_j+\omega_m)}. \quad \text{(A.3}e\text{)}$$

The coefficients in equations (18*a*) and (18*b*) can be simply obtained with the substitution $K_{1,m}\to -K_{1,m}$, $K_{2,m}\to -K_{2,m}$, $\omega_m\to -\omega_m$ in equations (A.3*a*)–(A.3*e*).

The coefficients in equations (20*a*)–(20*c*) are given by

$$a_j=\frac{1}{d_j}\left(\frac{K_{1,j}^2B_j}{\omega_j(1+K_j^2)}-K_{1,j}C_j-K_{2,j}D_j-\frac{\omega_jK_{2,j}^2B_j}{(1+K_j^2)(\Omega^2-\omega_j^2)}\right) \quad \text{(A.4}a\text{)}$$

$$b_j=\frac{1}{d_j}\left(\frac{K_{1,j}K_{2,j}(\omega_jD_j+\mathrm{i}\Omega E_j)}{\Omega^2-\omega_j^2}-K_{1,j}-\frac{K_{1,j}^2C_j}{\omega_j}\right) \quad \text{(A.4}b\text{)}$$

and if $j\neq l$,

$$c_{jl}=-\sum_{l=1(l\neq j)}^{N}(K_{1,j}(C_{1,jm}C_m+D_{1,jm}C_m+C_{2,jm}+D_{2,jm})$$
$$+K_{2,j}(C_{1,jm}D_m+D_{1,jm}D_m+C_{4,jm}+D_{4,jm}))$$
$$-\frac{K_{1,j}}{\omega_j}(K_{1,j}(C_{2,jm}C_m+D_{2,jm}D_m)+K_{1,m}(C_mC_{4,jm}-C_mD_{4,jm})$$
$$+(K_{1,j}+K_{1,m})D_mC_{2,jm}+(K_{1,j}-K_{1,m})D_mD_{2,jm})$$

if $j = l$,

$$c_{jl} = \frac{1}{d_j}\Bigg(- K_{1,j}(C_j B_{1,j} + B_{3,j}) - K_{2,j}\Bigg(B_{1,j}D_j + B_{4,j} - \frac{2\mathrm{i}E_j}{\Omega}(K_{1,j}C_j + K_{2,j}D_j)\Bigg)$$
$$+\Bigg(\frac{K_{1,j}^2}{\omega_j(1+K_j^2)} + \frac{K_{2,j}^2\omega_j B_j}{(\omega_j^2-\Omega^2)(1+K_j^2)}\Bigg)(-B_j^2 + B_{2,j})$$
$$-\frac{K_{1,j}}{\omega_j}\Bigg(K_{1,j}B_{3,j}C_j - \frac{2\mathrm{i}E_j K_{2,j}C_j}{\Omega}(K_{1,j}C_j + K_{2,j}D_j)$$
$$+2K_{2,j}D_j B_{3,j} - K_{2,j}B_{4,j}C_j\Bigg)$$
$$+\frac{K_{2,j}\omega_j}{\Omega^2-\omega_j^2}\Bigg(K_{2,j}D_j\Bigg(B_{4,j} - \frac{2\mathrm{i}E_j}{\Omega}(K_{1,j}C_j + K_{2,j}D_j)\Bigg)$$
$$-K_{1,j}B_{3,j}D_j + 2K_{1,j}C_j B_{4,j}\Bigg) + \frac{\mathrm{i}K_{2,j}\Omega}{\Omega^2-\omega_j^2}\Bigg(K_{1,j}(2C_j B_{5,j} + E_j B_{3,j})$$
$$+K_{2,j}\Bigg(2B_{5,j}D_j - \frac{2E_j^2}{\Omega}(K_{1,j}C_j + K_{2,j}D_j) + B_{4,j}E_j\Bigg)\Bigg)\Bigg) \tag{A.4c}$$

where

$$d_j = 1 + \frac{K_{1,j}C_j}{\omega_j} - \frac{D_j K_{2,j}\omega_j + \mathrm{i}K_{2,j}E_j\Omega}{\Omega^2-\omega_j^2} \tag{A.4d}$$

and

$$f_j = C_j^2 - B_j^2 \tag{A.4e}$$

$$g_j = U_{1,j}\left(2D_j - \frac{2\mathrm{i}E_j}{\Omega}(K_{1,j}C_j + K_{2,j}D_j)\right) \tag{A.4f}$$

$$h_j = U_{2,j}\left(2D_j - \frac{2\mathrm{i}E_j}{\Omega}(K_{1,j}C_j + K_{2,j}D_j)\right) \tag{A.4g}$$

$$\tilde{f}_j = 2C_j + U_{1,j}(C_j^2 - B_j^2) \tag{A.4h}$$

$$\tilde{g}_j = 2D_j + U_{2,j}(C_j^2 - B_j^2) - \frac{2\mathrm{i}E_j}{\Omega}(K_{1,j}C_j + K_{2,j}D_j). \tag{A.4i}$$

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